Quantum Control Theory;
The basics

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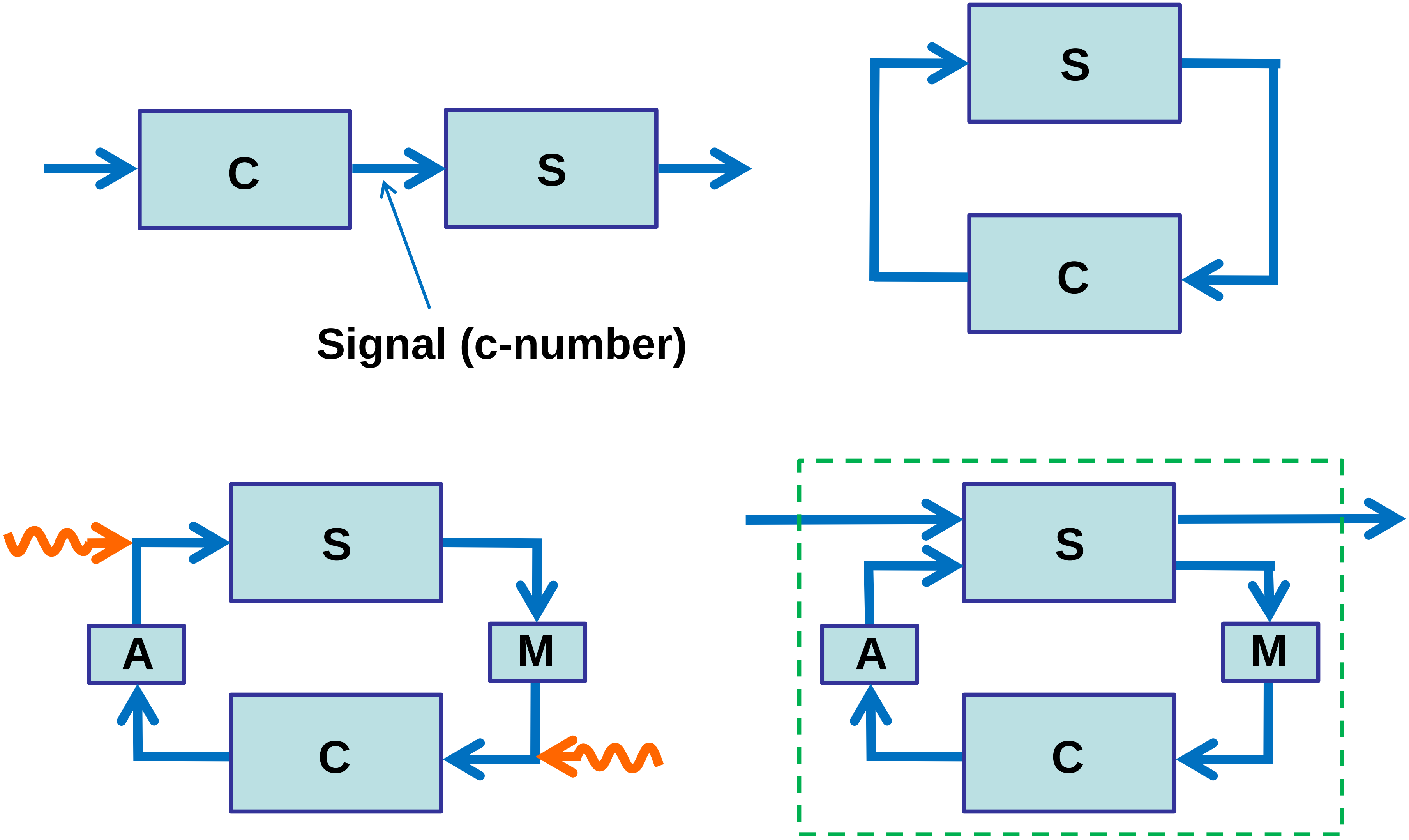
Contents

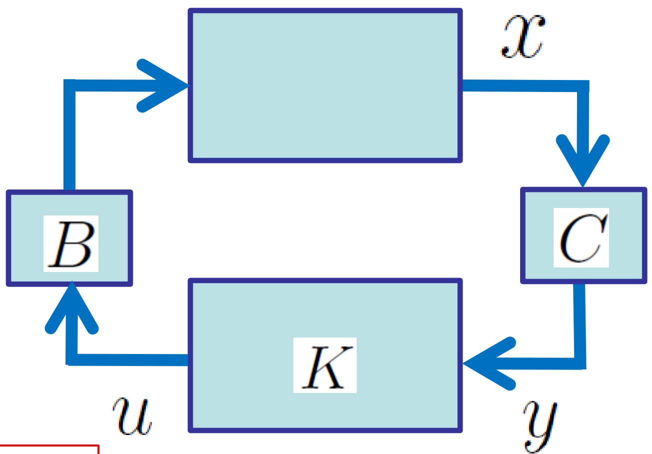
- **Classical control theory**
- **Quantum control theory --- Continuous meas.**

1. C control (1) : Various systems and purposes



1. C control (2) : Linear feedback control --- stabilization

Linear system $\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}$



Target value : $x = 0$

Feedback control law : $u = Ky$

→ **Design** K **so that the closed-loop system** $\dot{x} = (A + BKC)x$ **is stable.**

(e.g.) $\begin{cases} \dot{x} = 2x + u \\ y = x \end{cases}$ **is unstable when** $u = 0$ **;** $x_t = e^{2t} x_0 \to \infty$

Via the FB law $u = ky$**, the system becomes** $\dot{x} = (2 + k)x$

Thus choosing $k = -3$**, the system is stabilized ;** $x_t = e^{-t} x_0 \to 0$

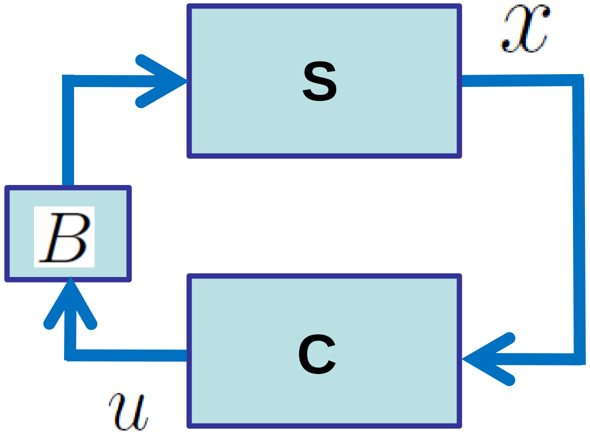
1. C control (3) : Optimal linear feedback control

Linear system $\dot{x} = Ax + Bu$

Suppose the input u **is a function of** x

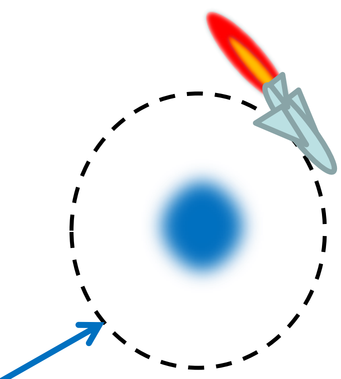
Control purpose :

$$J_T = \frac{1}{2}\int_0^T \left(x_t^T P x_t + u_t^T Q u_t\right) dt \rightarrow \min.$$



FB control law that minimizes the above cost function is given by :

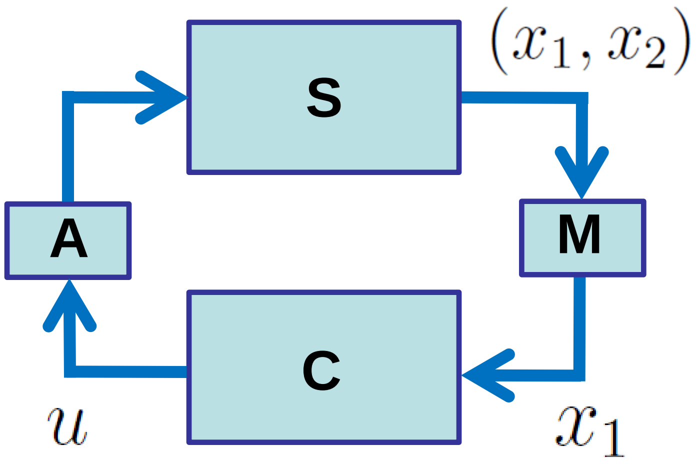
$$\begin{cases} u_t^* = -B^T K_t x_t \\ \dot{K}_t = -K_t A - A^T K_t + K_t B B^T K_t - P \end{cases}$$



Target trajectory : $x = 0$

1. C control (4) : Nonlinear FB control --- Lyapunov method

A nonlinear system $\begin{cases} \dot{x}_1 = x_2^2 + u \\ \dot{x}_2 = -x_1 x_2 \end{cases}$



Suppose u **is a function of** x_1

Control purpose : $(x_1, x_2) \to (0, 0)$

Set a non-negative function $V(x_1, x_2) = x_1^2 + x_2^2 \geq 0$ **then we have**

$$\dot{V} = 2x_1(x_2^2 + u) + 2x_2(-x_1 x_2) = 2ux_1$$

→ **FB control law** $u = -x_1$ **yields** $\dot{V} = -2x_1^2 \leq 0$

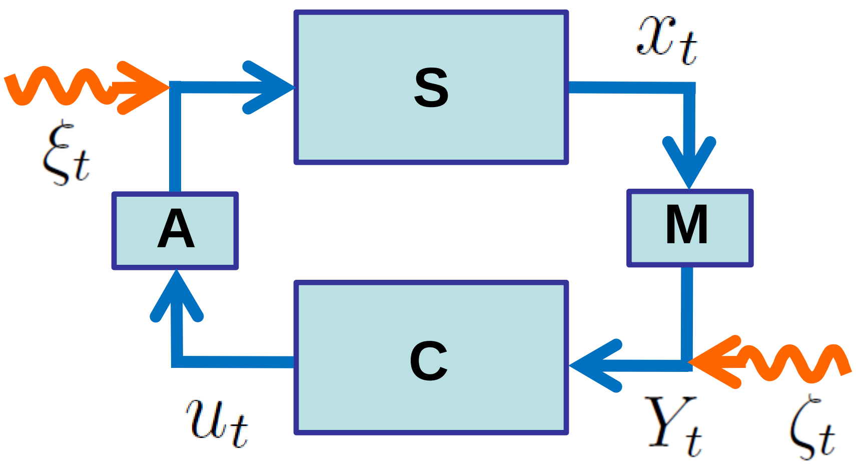
Thus V **always decreases in time.**

→ **In particular, we have** $(x_1, x_2) \to (0, 0)$

1. C control (5) : Stochastic FB control

Stochastic system

$$\begin{cases} \dot{x}_t = f(x_t) + g_1(x_t)u_t + g_2(x_t)\xi_t \\ Y_t = h(x_t) + \zeta_t \end{cases}$$



Before discussing how to control…

- **White noise approximation:** $\mathbb{E}(\xi_s\xi_t) = \delta(s-t)$

- **Noise added in** $[t,\ t+dt]$ **= Wiener increment** dW_t

  Formally, $\xi_t = \dfrac{dW_t}{dt}$ **and subjected to** $\mathcal{N}(0, dt)$ **hence** $dW_t^2 = dt$

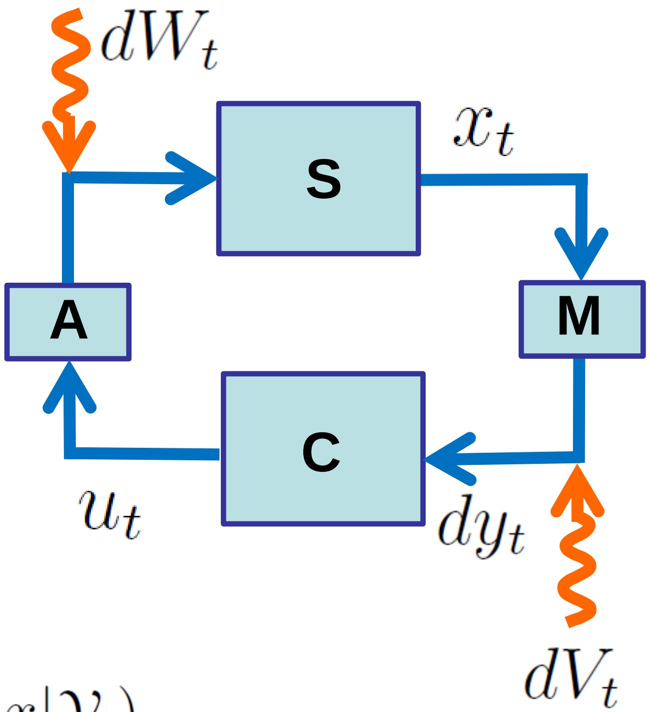
- **Dynamics in** $[t,\ t+dt]$ **= stochastic differential equation (SDE)**

$$\begin{cases} dx_t = f(x_t)dt + g_1(x_t)u_t dt + g_2(x_t)dW_t \\ dy_t = h(x_t)dt + dV_t \end{cases}$$

1. C control (5) : Stochastic FB control

Stochastic system

$$\begin{cases} dx_t = f(x_t)dt + g_1(x_t)u_t dt + g_2(x_t)dW_t \\ dy_t = h(x_t)dt + dV_t \end{cases}$$



It would be a good idea to use the estimate value of x_t, say $\pi_t(x)$, and design an estimate-based FB control

$$u = F(\pi(x))$$

→ **Need the conditional probability :** $p_t(x|\mathcal{Y}_t)$

Actually, $\pi_t(x) = \int x p_t(x|\mathcal{Y}_t)dx$

We want to have the quantum version of this scheme.

2. Q control (1) : Prelimi (i) Conditional prob. & Measurement

Classical conditional probability : Dice as an example

$$\Omega = \{1,2,3,4,5,6\} \quad \mathbb{P} = \{p_1, p_2, p_3, p_4, p_5, p_6\}$$ **(i.e.,** $\mathbb{P}(k) = p_k$ **)**

$$\mathbb{P}(k|\mathrm{even}) = \frac{\mathbb{P}(k,\mathrm{even})}{\mathbb{P}(\mathrm{even})} = \begin{cases} 0 \\ p_2/(p_2+p_4+p_6) \\ 0 \\ p_4/(p_2+p_4+p_6) \\ 0 \\ p_6/(p_2+p_4+p_6) \end{cases}$$

Prob. distribution conditioned on the result of "even"

$$\mathbb{P}(k|\mathrm{odd}) = \frac{\mathbb{P}(k,\mathrm{odd})}{\mathbb{P}(\mathrm{odd})} = \begin{cases} p_1/(p_1+p_3+p_5) \\ 0 \\ p_3/(p_1+p_3+p_5) \\ 0 \\ p_5/(p_1+p_3+p_5) \\ 0 \end{cases}$$

Prob. distribution conditioned on the result of "odd"

2. Q control (1) : Prelimi (i) Conditional prob. & Measurement

Represent using quantum mechanics

- **state** $$\rho = \begin{bmatrix} p_1 & & & & & \\ & p_2 & & & & \\ & & p_3 & & & \\ & & & p_4 & & \\ & & & & p_5 & \\ & & & & & p_6 \end{bmatrix}$$

- **observable** $$A = (+1)\begin{bmatrix} 0 & & & & & \\ & 1 & & & & \\ & & 0 & & & \\ & & & 1 & & \\ & & & & 0 & \\ & & & & & 1 \end{bmatrix} + (-1)\begin{bmatrix} 1 & & & & & \\ & 0 & & & & \\ & & 1 & & & \\ & & & 0 & & \\ & & & & 1 & \\ & & & & & 0 \end{bmatrix} = (+1)E_1 + (-1)E_2$$

Projection onto "even" → E_1

Projection onto "odd" → E_2

2. Q control (1) : Prelimi (i) Conditional prob. & Measurement

State reduction = conditional probability

ρ → even →

$$\rho_{\text{even}} = \frac{E_1 \rho E_1}{\mathrm{Tr}\left(E_1 \rho\right)} = \frac{1}{p_2 + p_4 + p_6} \begin{bmatrix} 0 & & & & & \\ & p_2 & & & & \\ & & 0 & & & \\ & & & p_4 & & \\ & & & & 0 & \\ & & & & & p_6 \end{bmatrix}$$

ρ → odd →

$$\rho_{\text{odd}} = \frac{E_2 \rho E_2}{\mathrm{Tr}\left(E_2 \rho\right)} = \frac{1}{p_1 + p_3 + p_5} \begin{bmatrix} p_1 & & & & & \\ & 0 & & & & \\ & & p_3 & & & \\ & & & 0 & & \\ & & & & p_5 & \\ & & & & & 0 \end{bmatrix}$$

2. Q control (1) : Prelimi (ii) Generalized measurement

- **State preparation and interaction**

$$|\Phi\rangle_{AB} \rightarrow U_{AB}|\Phi\rangle_{AB}$$

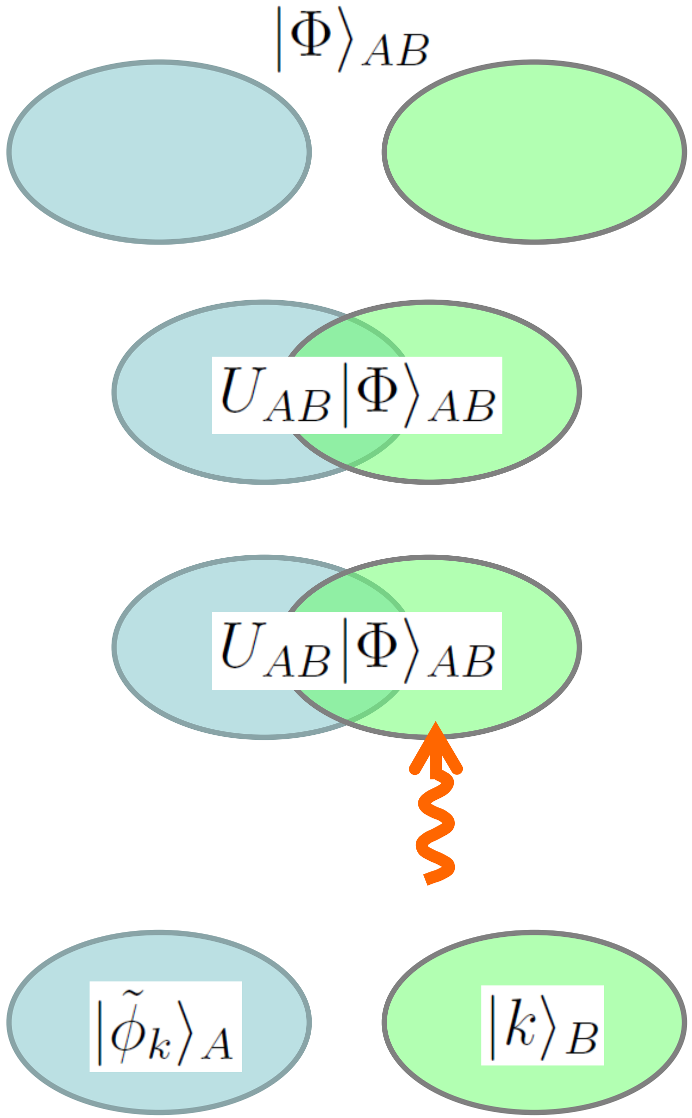
- **Projection measurement on the ancilla**

$$|\tilde{\Phi}_k\rangle_{AB} = (I_A \otimes |k\rangle_B\langle k|)U_{AB}|\Phi\rangle_{AB}$$
$$= \langle k|U_{AB}|\Phi\rangle_{AB} \otimes |k\rangle_B$$
$$= |\tilde{\phi}_k\rangle_A \otimes |k\rangle_B$$

--- **state reduction**

- **Output probability**

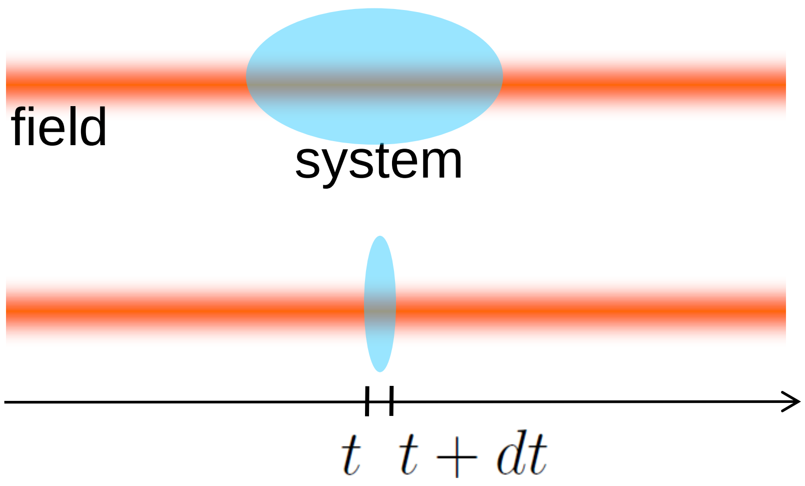
$$\mathbb{P}(k) = {}_{AB}\langle\tilde{\Phi}_k|\tilde{\Phi}_k\rangle_{AB} = {}_{A}\langle\tilde{\phi}_k|\tilde{\phi}_k\rangle_A$$



2. Q control (3) : Continuous meas. --- field as an ancilla

- **Interaction with the vacuum field**

$$H = i(cb_t^\dagger - c^\dagger b_t)$$
$$= i(c \otimes b_t^\dagger - c^\dagger \otimes b_t)$$



- **Quantum white noise**

$$[b_s, b_t^\dagger] = \delta(s-t)$$

- **Quantum Wiener increment** dB_t

Formally, $b_t = \dfrac{dB_t}{dt}$ **and the output prob. of** $dB_t + dB_t^\dagger$ **is** $\mathcal{N}(0, dt)$

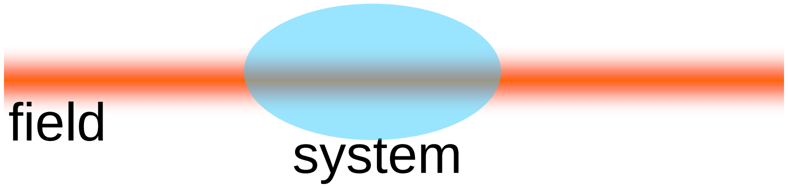
$$dB_t^2 = dB_t^{\dagger 2} = dB_t^\dagger dB_t = 0 \qquad dB_t dB_t^\dagger = dt$$

- **Interaction Hamiltonian in** $[t,\ t+dt]$

$$H dt = i(c dB_t^\dagger - c^\dagger dB_t)$$
$$= i(c \otimes dB_t^\dagger - c^\dagger \otimes dB_t)$$

2. Q control (3) : Continuous meas. --- interaction

- **Interaction Unitary in** $[t,\ t+dt]$



$$
\begin{aligned}
U_{t,t+dt} &= \mathrm{e}^{-iHdt} \\
&= \exp(c\,dB_t^\dagger - c^\dagger dB_t) \\
&= I + c\,dB_t^\dagger - c^\dagger dB_t - \frac{1}{2}c^\dagger c\,dt
\end{aligned}
$$

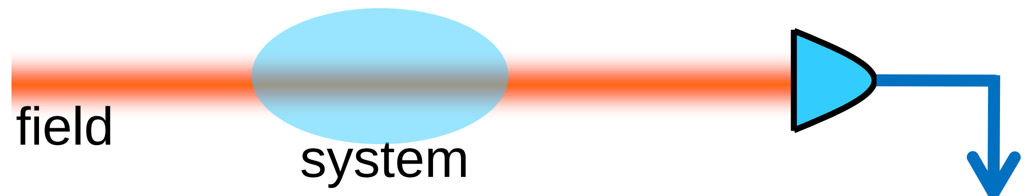
- **interaction**

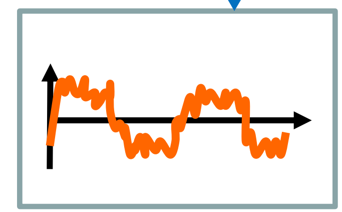
$$
\begin{aligned}
|\phi(t)\rangle|0\rangle \rightarrow |\Phi(t+dt)\rangle &= U_{t,t+dt}|\phi(t)\rangle|0\rangle \\
&= \left[I + c\,dB_t^\dagger - c^\dagger dB_t - \frac{1}{2}c^\dagger c\,dt\right]|\phi(t)\rangle|0\rangle \\
&= \left[I + c\,dB_t^\dagger - \frac{1}{2}c^\dagger c\,dt\right]|\phi(t)\rangle|0\rangle \\
&= \left[I + c(dB + dB_t^\dagger) - \frac{1}{2}c^\dagger c\,dt\right]|\phi(t)\rangle|0\rangle
\end{aligned}
$$

$$dB_t|0\rangle = 0$$

2. Q control (3) : Continuous meas. --- projection meas.

- **Homodyne meas. of the field**
  ||
  Projection onto the quadrature basis





$$(b_t + b_t^\dagger)|Y\rangle = Y_t|Y\rangle \quad \textbf{(instantaneous value)}$$

$$(dB_t + dB_t^\dagger)|dy\rangle = dy_t|dy\rangle \quad \textbf{(increment value)}$$

- **State reduction ; The unnormalized ket vector is given by**

$$\begin{aligned}|\tilde{\phi}(t+dt)\rangle &= \langle dy|\Phi(t+dt)\rangle \\ &= \langle dy|\Big[I + c(dB + dB_t^\dagger) - \frac{1}{2}c^\dagger c dt\Big]|\phi(t)\rangle|0\rangle \\ &= \Big[I + c dy_t - \frac{1}{2}c^\dagger c dt\Big]|\phi(t)\rangle\langle dy|0\rangle\end{aligned}$$

Recall: $|\tilde{\Phi}_k\rangle_{AB} = (I_A \otimes |k\rangle_B\langle k|)U_{AB}|\Phi\rangle_{AB} = \langle k|U_{AB}|\Phi\rangle_{AB} \otimes |k\rangle_B = |\tilde{\phi}_k\rangle_A \otimes |k\rangle_B$

2. Q control (3) : Continuous meas. --- output probability

- **State reduction ; The unnormalized ket vector is:**

$$|\tilde{\phi}(t+dt)\rangle = \Big[I + c\,dy_t - \frac{1}{2}c^\dagger c\,dt\Big]|\phi(t)\rangle\underline{\langle dy|0\rangle}$$

Recall: $\langle x|0\rangle = \dfrac{1}{\sqrt{\pi}}e^{-x^2/2}$

- **Output probability**

$$\mathbb{P}(dy) = \langle\tilde{\phi}(t+dt)|\tilde{\phi}(t+dt)\rangle$$
$$= \frac{1}{\sqrt{\pi dt}}\exp\Big[-\frac{1}{2dt}(dy - \langle c + c^\dagger\rangle dt)^2\Big]$$

→ $dW = dy - \langle c + c^\dagger\rangle dt$ **is subjected to** $\mathcal{N}(0, dt)$

- **Summary : time evolution of the unnormalized ket vector**

$$\begin{cases} d|\tilde{\phi}(t)\rangle = \Big[-\dfrac{1}{2}c^\dagger c\,dt + c\,dy_t\Big]|\tilde{\phi}(t)\rangle \\ dy = \langle c + c^\dagger\rangle dt + dW \end{cases}$$

2. Q control (4) : Stochastic Schrodinger and Master Eqs.

- Time evolution of the normalized ket vector = **SSE**

$$|\phi\rangle = \frac{|\tilde{\phi}\rangle}{\sqrt{\langle\tilde{\phi}|\tilde{\phi}\rangle}} \quad \rightarrow \quad d|\phi(t)\rangle = \frac{|\tilde{\phi}(t+dt)\rangle}{\sqrt{\langle\tilde{\phi}(t+dt)|\tilde{\phi}(t+dt)\rangle}} - \frac{|\tilde{\phi}(t)\rangle}{\sqrt{\langle\tilde{\phi}(t)|\tilde{\phi}(t)\rangle}}$$

$$\rightarrow \boxed{d|\phi\rangle = \Big[-\frac{1}{2}(c^\dagger c - 2\langle x\rangle c + \langle x\rangle^2)dt + (c - \langle x\rangle)dW \Big] |\phi\rangle}$$

$$\langle x\rangle = \langle c + c^\dagger\rangle/2$$

- Time evolution of the density operator = **SME**

$$\rho = |\phi\rangle\langle\phi| \quad \rightarrow \quad d\rho(t) = |\phi(t+dt)\rangle\langle\phi(t+dt)| - |\phi(t)\rangle\langle\phi(t)|$$

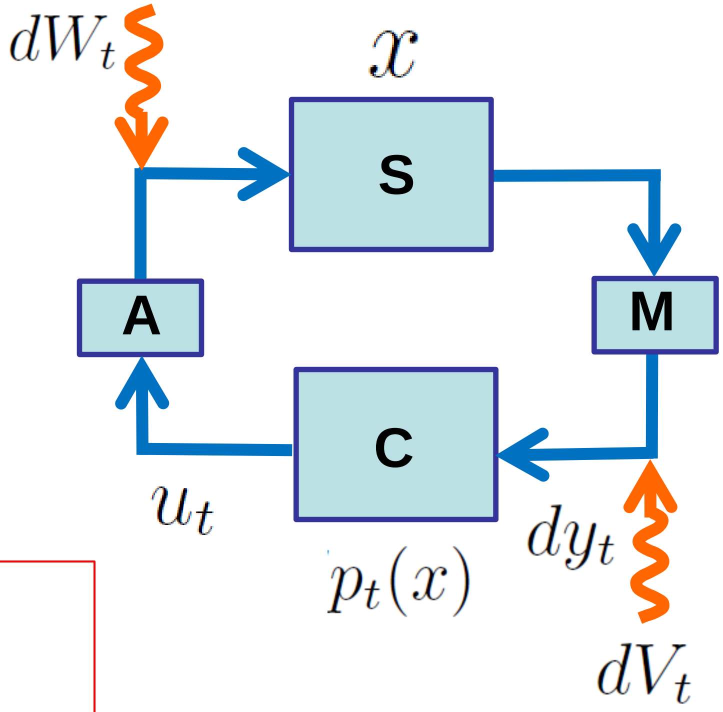
$$\rightarrow \boxed{d\rho = \mathcal{L}^*\rho dt + \big[c\rho + \rho c^\dagger - \langle c + c^\dagger\rangle\rho\big](dy - \langle c + c^\dagger\rangle dt)}$$

$$\mathcal{L}^*\rho := -i[H,\rho] + c\rho c^\dagger - \frac{1}{2}c^\dagger c\rho - \frac{1}{2}\rho c^\dagger c$$

2. Q control (5) : General framework (classical)

Stochastic system

$$\begin{cases} dx_t = f(x_t)dt + g_1(x_t)u_t dt + g_2(x_t)dW_t \\ dy_t = h(x_t)dt + dV_t \end{cases}$$



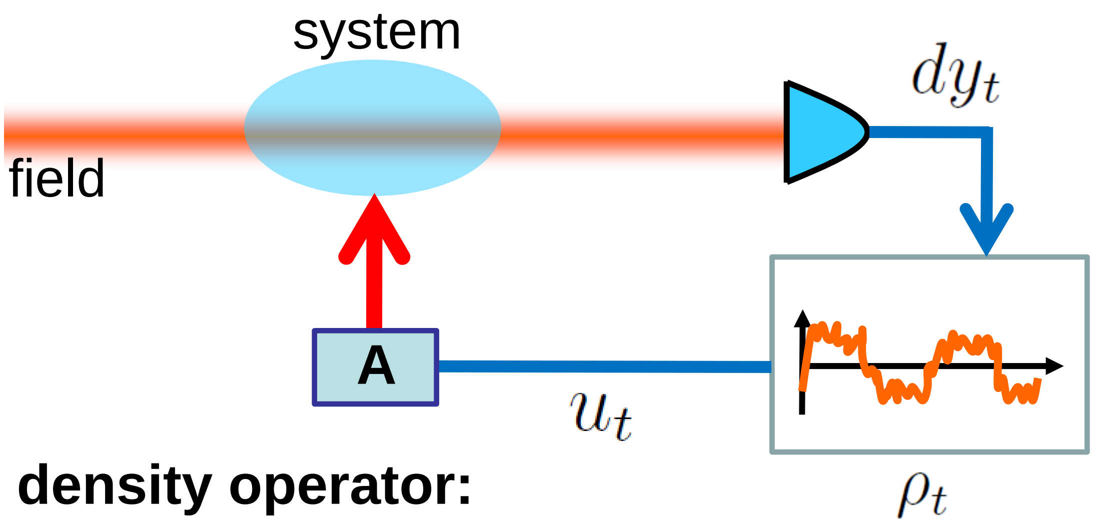
Time evolution of the conditional probability density $p_t(x) := p_t(x|\mathcal{Y}_t)$ **:**

$$dp_t(x) = \Big[-\frac{\partial(p_t f)}{\partial x} + \frac{1}{2}\frac{\partial(p_t g^2)}{\partial x^2} \Big] dt + p_t(x)[h(x) - \pi_t(h(x))][dy - \pi_t(h(x))dt]$$

→ **Via a state-dependent FB control** $u = F(p_t)$

We aim to attain a desirable state convergence: $p_t \to p^*$

2. Q control (5) : General framework (quantum)



Time evolution of the cond. density operator:

$$d\rho = \mathcal{L}^*\rho dt + \left[c\rho + \rho c^\dagger - \langle c + c^\dagger \rangle \rho\right](dy - \langle c + c^\dagger \rangle dt)$$

$$\mathcal{L}^*\rho := -i[H, \rho] + c\rho c^\dagger - \frac{1}{2}c^\dagger c\rho - \frac{1}{2}\rho c^\dagger c$$

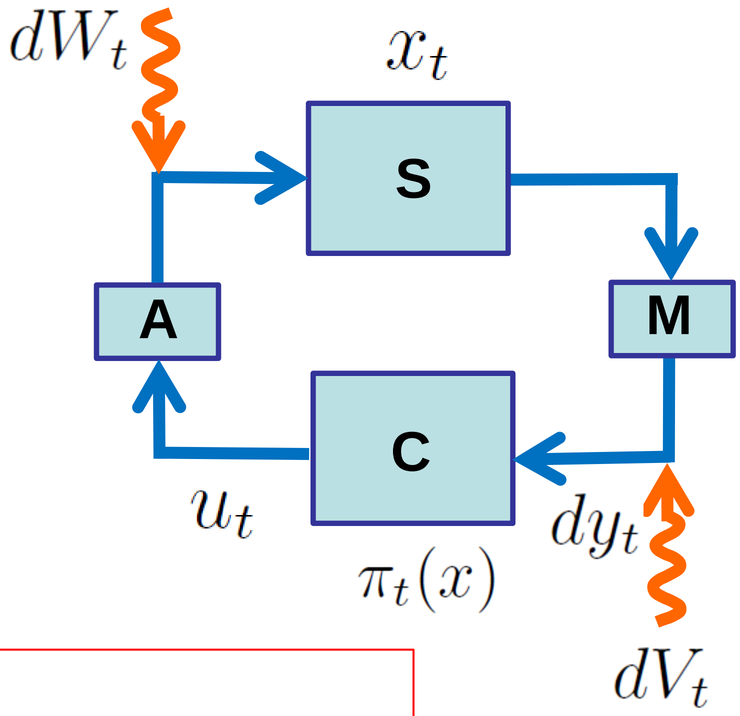
→ **Via a state-dependent FB control** $u = F(\rho_t)$

We aim to attain a desirable state convergence : $\rho_t \to \rho^*$

2. Q control (5) : General framework (C, Heisenberg pic.)

Classical stochastic system (SDE):

$$\begin{cases} dx_t = f(x_t)dt + g_1(x_t)u_t dt + g_2(x_t)dW_t \\ dy_t = h(x_t)dt + dV_t \end{cases}$$



Time evolution of the estimate of x_t is given by the following filtering equation

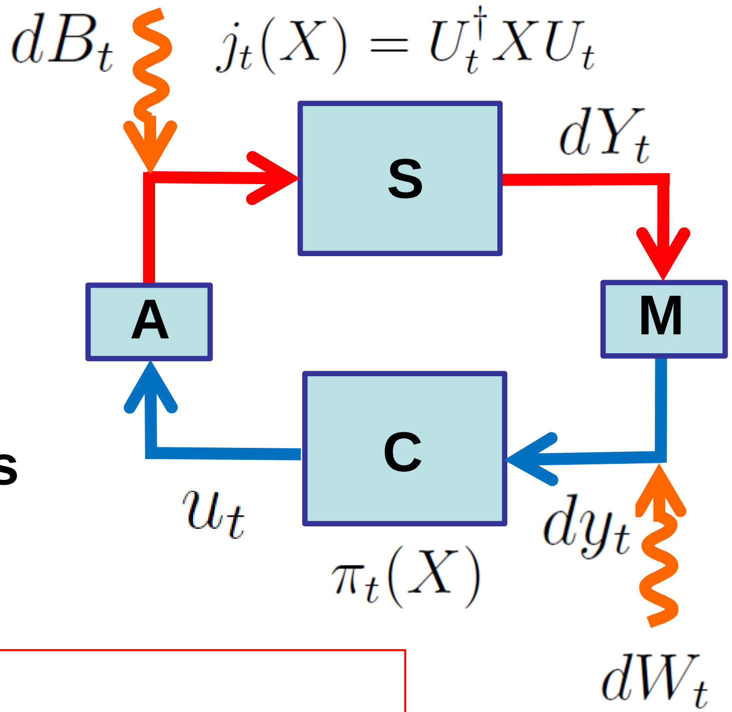
(use $\pi_t(x) = \int x p_t(x|\mathcal{Y}_t)dx$ **)**

$$\begin{aligned} d\pi_t(x) &= \pi_t(f(x))dt \\ &+ \big[\pi_t(xh(x)) - \pi_t(x)\pi_t(h(x))\big]\big[dy - \pi_t(h(x))dt\big] \end{aligned}$$

→ **Estimation-based FB controller** $u = F(\pi(x))$ **can be a solution to e.g. some optimal control problem:**

$$J_T = \frac{1}{2}\mathbb{E}\Big[\int_0^T (x_t^2 + ru_t^2)dt\Big] = \frac{1}{2}\mathbb{E}\Big[\int_0^T \Big(\pi_t(x^2) + ru_t^2\Big)dt\Big] \to \min.$$

2. Q control (5) : General framework (Q, Heisenberg pic.)



Quantum stochastic system (QSDE):

$$
\begin{cases}
dj_t(X) = j_t(\mathcal{L}X)dt \\
\qquad\qquad + j_t([X,c])dB^\dagger - j_t([X,c^\dagger])dB \\
dY_t = j_t(c+c^\dagger)dt + dB_t + dB_t^\dagger
\end{cases}
$$

Time evolution of the estimate of $j_t(X)$ is given by the following filtering equation (use $\pi_t(X) = \mathrm{Tr}\,(X\rho_t)$)

$$
\begin{aligned}
d\pi_t(X) &= \mathcal{L}X\,dt \\
&\quad + [\pi_t(Xc + c^\dagger X) - \pi_t(c+c^\dagger)\pi_t(X)](dy - \pi_t(c+c^\dagger)dt)
\end{aligned}
$$

→ **Estimation-based FB controller $u = F(\pi_t(X))$ can be a solution to e.g. some optimal control problem:**

$$
J_T = \frac{1}{2}\mathbb{E}\Big[\int_0^T (j_t(X)^2 + ru_t^2)dt\Big] = \frac{1}{2}\mathbb{E}\Big[\int_0^T \Big(\pi_t(X^2) + ru_t^2\Big)dt\Big] \to \min.
$$